

# Using the Internal Logic of a Topos Related to the Topos of Forests to Model Search Spaces for Problems

Fernando Náufel do Amaral      Edward Hermann Haeusler

From a logical point of view, one of the most interesting developments of topos theory [3] has been the investigation of the internal logics of topoi by means of local set theory (henceforth LST) [1]. This is accomplished by viewing any topos as a model of a theory in the language of LST, which is basically a higher-order language, or type theory. Higher-order logic has been very important in foundational studies in computer science. Our contribution focusses on an application of topos theory and LST to the field of (meta)heuristic search strategies [2], which attempt to solve a given computational problem by making use of “rules of thumb”, approximations, guesses and stochastic processes, sometimes inspired by physical (e.g. simulated annealing) or biological (e.g. genetic algorithms) phenomena. The present abstract describes how topos-theoretical tools and techniques can be employed to construct a structural model for (meta)-heuristic search spaces.

**Definition 1 (Problems).** *A problem is a triple  $P = \langle D, R, p \rangle$ , with  $D$  and  $R$  countable, nonempty sets, and  $p \subseteq D \times R$  a relation. Elements  $d \in D$  are called instances; elements  $r \in R$  are answers;  $(d, r) \in p$  means that  $r$  is a correct answer for instance  $d$ .*

**Definition 2 (Reductions).** *Given  $P = \langle D, R, p \rangle$  and  $P' = \langle D', R', p' \rangle$ , a reduction  $P \xrightarrow{(\tau, \sigma)} P'$  consists of a pair  $(\tau, \sigma)$  of functions computable in polynomial time, with  $\tau : D \rightarrow D'$  and  $\sigma : R' \rightarrow R$  such that correct answers are preserved; more precisely, for every  $d \in D$  and every  $r' \in R'$ , we have that  $(\tau(d), r') \in p' \Rightarrow (d, \sigma(r')) \in p$ .*

**Definition 3 (The category **Prob** of problems).** ***Prob** is a designated thin, skeletal category having as objects the set of problems of interest and having as morphisms the set of reductions of interest.*

The search spaces we will define for the problems in **Prob** are based on forests. A convenient category-theoretical definition of forests and forest homomorphisms is the following:

**Definition 4 (Forests, forest homomorphisms).** *A forest  $S$  is a functor  $S : \omega^{op} \rightarrow \mathbf{Set}$ , where  $\omega^{op}$  is the category  $0 \leftarrow 1 \leftarrow 2 \leftarrow \dots$ . A homomorphism  $h$  from a forest  $S$  to a forest  $S'$  is a natural transformation  $h : S \rightarrow S'$ .*

**Definition 5 (The category **Forest**).** *The category **Forest** of forests and forest homomorphisms is the functor category  $\mathbf{Set}^{\omega^{op}}$ .*

A search space for a problem  $P = \langle D, R, p \rangle$  will be an object  $FP$  of **Forest**, enriched with some additional information about  $P$ . More specifically, we want to label each vertex  $v$  of the forest  $FP$  with a set of answers of  $P$  (i.e., with a set  $\lambda(v)$  such that  $\lambda(v) \subseteq R$ ). This is done via a natural transformation to the contravariant functor defined below. (The meaning of the sets of answers labeling the vertices of the forest depends on the intended structure of the search space, and is not explained here.)

**Definition 6 (The  $L$  functor).**  $L : \mathbf{Prob}^{op} \rightarrow \mathbf{Forest}$  is the functor that maps a problem  $P = \langle D, R, p \rangle$  to the forest  $LP$  with infinitely many levels, whose set of vertices at level 0 is  $\wp(R)$  (the powerset of  $R$ ), and whose set of vertices at level  $i$ , for  $i > 0$ , is  $\wp(R)^{i+1}$ , the set of all  $(i+1)$ -tuples whose components are sets of answers in  $R$ . As for morphisms,  $L$  maps a reduction  $P \xrightarrow{(\tau, \sigma)} P'$  to the forest homomorphism  $LP' \xrightarrow{L(\tau, \sigma)} LP$  such that a vertex  $(A'_1, \dots, A'_i)$  at the  $i$ th level of  $LP'$  is mapped to the vertex  $(\sigma(A'_1), \dots, \sigma(A'_i))$  at the  $i$ th level of  $LP$ .

**Definition 7 (Answer forest assignment).** An answer forest assignment is a pair  $\langle F, \lambda \rangle$ , where  $F : \mathbf{Prob}^{op} \rightarrow \mathbf{Forest}$  is a functor and  $\lambda : F \rightarrow L$  is a natural transformation.

Take a problem  $P$  in **Prob**; to see how  $\lambda_P$  corresponds to a labeling of the vertices of  $FP$ , note that a vertex  $v$  at the  $i$ th level of  $FP$  with  $\lambda(v) = (A_1, \dots, A_i)$  is considered to be labeled by the set  $A_i$ . The other components  $A_1, \dots, A_{i-1}$  store the labels of the ancestor vertices of  $v$  from the root down to the parent of  $v$ . Furthermore, because  $\lambda$  is a natural transformation, the labeling of the forests assigned to different problems in **Prob** is done consistently (i.e., naturally).

**Proposition 1.** The collection of all answer forest assignments is the collection of objects of the slice category  $(\mathbf{Forest}^{\mathbf{Prob}^{op}}) \downarrow L$ , where  $\mathbf{Forest}^{\mathbf{Prob}^{op}}$  is the category of all contravariant functors from **Prob** to **Forest** and  $L$  is the functor introduced in Def. 6.

**Theorem 1.** Let this category of answer forest assignments be called **AFA**. The category **AFA** is a topos.

Each object of **AFA** corresponds, thus, to an assignment of search spaces to the problems in **Prob**. These assignments are related via the morphisms of **AFA**. Theorem 1 is important because it allows us to describe these assignments and relationships in the internal logic of the topos **AFA**, using the language of Local Set Theory (LST). Furthermore, as explained elsewhere, we have used LST not only to describe search spaces assigned to problems, but also to define different search strategies to traverse these spaces in a systematic way.

## References

- [1] J. L. Bell. *Toposes and Local Set Theories, an Introduction*. Oxford University Press, 1988.
- [2] F. Glover and G.A. Kochenberger, editors. *Handbook of Metaheuristics*. Kluwer, 2002.
- [3] R. Goldblatt. *Topoi – the Categorical Analysis of Logic*. North Holland, 1979.